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Spin flips in cyclotron emission by an electron

D B Melrose and K Russell

School of Physics, University of Sydney, NEW 2006, Australia

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Abstract

The spin dependence of cyclotron emission is treated using the non-relativistic limit of the Dirac equation; the Schrödinger–Pauli theory is inadequate because of the importance of spin–orbit coupling, which is an intrinsically relativistic effect. Only the choice of the magnetic moment as the spin operator is physically acceptable; all other spin operators precess at a rate comparable with or in excess of cyclotron transition rates. The spin-flip ($s = 1 \to -1$) transition rate is smaller than the non-spin-flip of the order B/B_c ($B_c = 4.4 \times 10^9$ T), and the reverse spin-flip ($s = -1 \to +1$) transition rate is smaller by a further factor of order $(B/B_c)^2$, implying that it is strongly forbidden. It is shown that there is a preference for electrons with spin $s = 1$ initially in a high Landau level, $n \gg 1$, to relax to the ground state, $s = -1$, $n = 0$, by stepwise jumps to the lowest Landau level for $s = 1$ and then making the spin-flip transition to $s = -1$, rather than making the spin-flip transition from a higher Landau level, and that this preference increases with decreasing B/B_c.

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1. Introduction

A fully relativistic quantum theory for gyromagnetic emission has been available since the 1960s, and it has been used to discuss synchrotron radiation in detail by Sokolov and Ternov (1968, 1986). These authors showed that synchrotron radiation tends to polarize electrons: spin flips $s = 1 \to -1$ are more probable than reverse spin flips $s = -1 \to 1$ and tend to cause the highly relativistic electrons to become 96% polarized in the state $s = -1$. However, this applies only if the spin operator with eigenvalues labelled $s = \pm 1$ is identified as the component of the magnetic moment operator along the magnetic field, B, which Sokolov and Ternov referred to as transverse polarization. In contrast, if the spin operator is identified as the component of the helicity operator along the magnetic field, referred to as longitudinal polarization by Sokolov and Ternov, there is no tendency for the electrons to become polarized as a result of synchrotron emission. These results apply in the synchrotron limit where the perpendicular momentum is highly relativistic. The perpendicular momentum is quantized, $p_n = (2neB)^{1/2} = m(2nB/B_c)^{1/2}$, where $n \geqslant 0$ is the Landau quantum number and

$B_c = m^2/e = 4.4 \times 10^9$ T is the critical magnetic field. (Natural units, with $\hbar = c = 1$ are used in this paper.) The synchrotron regime corresponds to $n \gg (2B/B_c)^{-1}$. In this paper we refer to the opposite limit, $n \ll (2B/B_c)^{-1}$, as the cyclotron regime.

A non-relativistic quantum theory for cyclotron emission is based on the Schrödinger–Pauli theory, which implies that the cyclotron motion is quantized as a simple harmonic oscillator, $\left(l + \frac{1}{2}\right)\Omega_e$ with $l = 0, 1, \ldots$ the orbital quantum number, with the inclusion of the spin, $\frac{1}{2}s\Omega_e$, leading to a perpendicular energy $p_n^2/2m = n\Omega_e$, with $n = l + \frac{1}{2}(1 + s)$ as the Landau quantum number. Cyclotron emission by non-relativistic electrons in a field $B \ll B_c$ occurs at close to harmonics of the cyclotron frequency: a transition from $n \to n'$ produces a photon with a frequency close to $j\Omega_e$, $j = n - n'$. An expansion in B/B_c implies that to lowest order only non-spin-flip transitions at the fundamental $j = 1$ occur; non-spin-flip transitions with $j \geqslant 2$ occur with a transition rate smaller by a factor of order $(B/B_c)^{j-1}$ than for $j = 1$. A spin-flip transition has a transition rate smaller than that for the corresponding non-spin-flip transition by a factor of order B/B_c (Melrose and Zheleznyakov 1981). The generalization to relativistic quantum theory has been discussed in detail (Pavlov *et al* 1980, Bezchastnov and Pavlov 1988) only in the case where the spin is neglected by averaging over the spin states.

In this paper we concentrate on one aspect of cyclotron emission that has not been treated adequately in the available literature: spin-flip transitions. One point that we emphasize is that the concept of a spin-flip transition is well defined only if the appropriate spin operator is chosen. The appropriate choice is to identify the spin as the magnetic moment operator, specifically, the component, $\hat{\mu}_z$, of the magnetic moment operator along B. Sokolov and Ternov (1968, 1986) argued for this based on the requirements that the spin operator commutes not only with the Dirac Hamiltonian, but also with the radiative correction to it; Herold *et al* (1982) argued that this choice of spin operator diagonalizes the operator corresponding to gyromagnetic emission. Apart from the ground state, $n = 0$, $l = 0$, s $= -1$, the electron states are doubly degenerate, and different choices of spin correspond to different mixtures of l, s in the two degenerate energy eigenstates for given n. This makes the concept of a 'non-spin-flip' or a 'spin-flip' transition dependent on the choice of spin operator. In order to treat spin-flip transitions correctly, one needs to use the non-relativistic approximation to Dirac theory, rather than the Schrödinger–Pauli theory; spin–orbit coupling, which is an intrinsically relativistic effect, cannot be neglected in the non-relativistic limit when considering spin-dependent effects. We illustrate this point by making a different choice of spin operator (the helicity) and showing that 'spin-flip' transitions in this case must be interpreted differently. A second point that we make, for the first time, is that with the appropriate choice of spin operator, the transition rate for reverse spin-flip transitions, $s = -1 \to 1$, is negligible compared with the rate for direct spin flips, $s = 1 \to -1$. Specifically, we show that the rate for reverse spin-flip transitions is smaller by a factor of order $(B/B_c)^3$ than that for the non-spin-flip transition. A third point that we discuss is a possible counterpart for cyclotron emission of Sokolov and Ternov's result that electrons become highly polarized as a result of synchrotron emission. Specifically, we consider the question as to how electrons initially at high l with $s = 1$ relax to the ground state, $l = 0$, $s = -1$ $(n = 0)$: is the spin-flip $s = 1 \to -1$ made preferentially at $l > 0$, or do the electrons preferentially relax to $l = 0$, $s = 1$ $(n = 1)$ before the spin-flip?

In section 2 we present the exact theory for gyromagnetic emission, and emphasize the importance of the correct (for this purpose) choice of spin operator. In section 3 we use the non-relativistic limit of the general theory to treat cyclotron emission, calculating the transition rates between eigenstates of $\hat{\mu}_z$ for both non-spin-flip and spin-flip transitions, and showing that reverse spin-flip transitions are of much higher order in B/B_c. In section 4 we repeat the calculation for the helicity eigenstates and show that 'spin flips' in this case are of the same order (in B/B_c) as non-spin-flip transitions between the eigenstates of $\hat{\mu}_z$. In section 5 we discuss whether spin flips occur preferentially at $l > 0$ or at $l = 0$. Our results are discussed in section 6.

2. Quantum theory of gyromagnetic emission

In this section we write down general expressions for gyromagnetic emission by an electron assuming that the spin operator is chosen to be the component, $\hat{\mu}_z$, of the magnetic moment operator, $\hat{\boldsymbol{\mu}}$.

2.1. Eigenstates of the magnetic moment operator

The explicit form for the wavefunction for a relativistic electron in a magnetic field depends on the choice of spin operator, and also on the choices of the representation of the Dirac algebra and the gauge for the vector potential, A, for the magnetic field. The wavefunction, which is a column matrix with four components, can be written as the product of a square matrix, which contains all the spacetime and gauge dependence, and a column matrix which contains all the spin dependence (Ritus 1970, 1972, Parle 1987). The general form is written down in appendix A.

The magnetic moment operator is (Sokolov and Ternov 1968)

$$\hat{\boldsymbol{\mu}} = m\boldsymbol{\sigma} - \mathrm{i}\gamma \times (\hat{\boldsymbol{p}} + e\boldsymbol{A}) \tag{1}$$

with the standard representation of the 4×4 matrices $\boldsymbol{\sigma}$ and γ adopted here. The eigenvalues of the z-component of the operator (1) are $s\varepsilon_n^0$, $s = \pm 1$

$$\varepsilon_n^0 = \left(m^2 + p_n^2\right)^{1/2} = \left(\varepsilon_n^2 - p_z^2\right)^{1/2} \qquad \varepsilon_n = \left(m^2 + p_z^2 + p_n^2\right)^{1/2} \tag{2}$$

where $\epsilon\varepsilon_n$ with $\epsilon = \pm 1$ are the eigenvalues of the Dirac Hamiltonian. In appendix A a factorized form (A.2) of the wavefunction is used to separate the spacetime and gauge-dependent part, $\mathcal{V}_g^\epsilon(\boldsymbol{x}, n, \epsilon p_z)$, from the spin-dependent part, $\varphi_s^\epsilon(n, \epsilon p_z)$. The spin-dependent part satisfies the eigenvalue equation (A.5) for the Hamiltonian for any choice of spin, with the solutions being doubly degenerate for $n > 0$. Simultaneous eigenfunctions of the Dirac Hamiltonian and of the magnetic moment are found by requiring that the spin-dependent part also be an eigenvalue of the z-component of the operator (1). Specifically, the magnetic moment eigenstates satisfy the eigenvalue equation

$$\begin{pmatrix} m & 0 & 0 & -\mathrm{i}p_n \\ 0 & -m & -\mathrm{i}p_n & 0 \\ 0 & \mathrm{i}p_n & m & 0 \\ \mathrm{i}p_n & 0 & 0 & -m \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = s\varepsilon_n^0 \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} \tag{3}$$

where C_1, C_2, C_3, C_4 denote the four components of $\varphi_s^\epsilon(n, \epsilon p_z)$. Simultaneous eigenfunctions of the Hamiltonian and μ_z are

$$\varphi_s^\epsilon(n, \epsilon p_z) = \frac{1}{\left(2\varepsilon_n^0 2\varepsilon_n V\right)^{1/2}} \begin{pmatrix} \lambda_{\epsilon s}\beta_s \\ -\mathrm{i}sP\lambda_{-\epsilon s}\beta_{-s} \\ P\lambda_{-\epsilon s}\beta_s \\ \mathrm{i}s\lambda_{\epsilon s}\beta_{-s} \end{pmatrix} \tag{4}$$

$$\lambda_\pm = \left(\varepsilon_n \pm \varepsilon_n^0\right)^{1/2} \qquad \beta_s = \left(\varepsilon_n^0 + sm\right)^{1/2}$$

where the identities $p_z = \lambda_{\epsilon s}\lambda_{-\epsilon s}$, $p_n = \beta_s\beta_{-s}$ are used with $P = p_z/|p_z|$, and where the normalization is to one charge in the volume V. The phase of each eigenfunction is arbitrary, and a specific choice (including the signs P, ϵ) is made to write (4) in a concise manner.

2.2. General theory for gyromagnetic emission

The general theory for gyromagnetic emission may be developed in terms of the probability per unit time that an electron in the state p_z, n, s emits a photon with frequency ω and (unimodular) polarization vector $\boldsymbol{e}$, in the range $\mathrm{d}^3\boldsymbol{k}/(2\pi)^3$ at wavevector $\boldsymbol{k}$, with transition to the state p'_z, n', s' (Tsytovich 1972, Melrose and Parle 1983). For emission by an electron ($\epsilon = \epsilon' = 1$) *in vacuo*, this probability is

$$w_{n,n',s,s'}(\boldsymbol{k}) = \frac{\mu_0 e^2}{2\omega}|\boldsymbol{e}^* \cdot \boldsymbol{\Gamma}_{n,n',s,s'}(\boldsymbol{k})|^2 2\pi\delta(\varepsilon_n - \varepsilon'_{n'} - \omega) \tag{5}$$

with conservation of parallel momentum implicit in $p'_z = p_z - k_z$ and with $\varepsilon_n = \left(m^2 + p_z^2 + 2neB\right)^{1/2}$ and $\varepsilon'_{n'} = \left(m^2 + p'^2_z + 2n'eB\right)^{1/2}$. The vertex function is defined in appendix B. For the magnetic moment eigenstates (4) it is

$$\begin{aligned}\boldsymbol{\Gamma}_{n,n',s,s'}(\boldsymbol{k}) = b'^*_{n'} b_n \big(&-s'\left[\lambda'_{s'}\lambda_s - P'P\lambda'_{-s'}\lambda_{-s}\right]\left[\beta'_{-s'}\beta_s \mathrm{e}^{-\mathrm{i}\psi} J^{n-1}_{n'-n+1} + s's\beta'_{s'}\beta_{-s}\mathrm{e}^{\mathrm{i}\psi} J^{n}_{n'-n-1}\right],\\ &-\mathrm{i}s'\left[\lambda'_{s'}\lambda_s - P'P\lambda'_{-s'}\lambda_{-s}\right]\left[\beta'_{-s'}\beta_s \mathrm{e}^{-\mathrm{i}\psi} J^{n-1}_{n'-n+1} - s's\beta'_{s'}\beta_{-s}\mathrm{e}^{\mathrm{i}\psi} J^{n}_{n'-n-1}\right],\\ &P\left[\lambda'_{s'}\lambda_{-s} + P'P\lambda'_{-s'}\lambda_s\right]\left[\beta'_{s'}\beta_s J^{n-1}_{n'-n} + s's\beta'_{-s'}\beta_{-s} J^{n}_{n'-n}\right]\big)\end{aligned} \tag{6}$$

$$b_n = \frac{(\mathrm{i}\mathrm{e}^{\mathrm{i}\psi})^n}{\left(2\varepsilon^0_n 2\varepsilon_n\right)^{1/2}} \qquad \lambda_s = \left(\varepsilon_n + s\varepsilon^0_n\right)^{1/2} \qquad \beta_s = \left(\varepsilon^0_n + sm\right)^{1/2}$$

with $P = p_z/|p_z|$, and similarly for the primed quantities. In (6) the wave vector is written $\boldsymbol{k} = (k_\perp \cos\psi, k_\perp \sin\psi, k_z)$, and one is free to choose $\psi = 0$ by rotating the axes so that $\boldsymbol{k}$ is in the x–z plane. The J-functions are related to generalized Laguerre polynomials

$$J^n_\nu(x) = (-)^\nu J^{n+\nu}_{-\nu}(x) = \left(\frac{n!}{(n+\nu)!}\right)^{1/2} \mathrm{e}^{-x/2} x^{\nu/2} L^\nu_n(x) \tag{7}$$

with argument $x = k_\perp^2/2eB$. It is implicit that the function $J^n_\nu(x)$ is identically zero for negative n.

3. Quantum theory of cyclotron emission

In this section the non-relativistic approximation is made to the general theory of section 2, and this is used to treat cyclotron emission between eigenstates of the magnetic moment operator, $\hat{\mu}_z$.

3.1. The cyclotron approximation

In treating cyclotron emission, the non-relativistic approximation is made in evaluating the vertex function. An important simplification results for $x = k_\perp^2/2eB \ll 1$, that is, when the argument of the J-functions in the vertex function (6) is small. One has

$$x = \frac{k_\perp^2}{2eB} = \frac{k^2 \sin^2\theta}{2m\Omega_e} = \frac{1}{2}\left(\frac{k_\perp}{m}\right)^2 \left(\frac{B}{B_c}\right)^{-1}. \tag{8}$$

For $k_\perp^2/2eB \ll 1$ the power series expansions of the J-functions converge rapidly, and each function may be approximated by its leading term (for $j > 0$):

$$J^n_{-j}(x) \approx \frac{(-)^j}{j!}\left(\frac{n!}{(n-j)!}\right)^{1/2} x^{j/2}. \tag{9}$$

Hence, when making the cyclotron approximation, one need retain only the terms with the smallest value of $|\nu|$ in J^n_ν.

3.2. Nonrelativistic approximation to the vertex function

Here the foregoing approximations are applied to the vertex function (6). This involves assuming that $p_z/m,\ p_n/m$ are small and their powers are expanding. The momentum, k_z, of the wave quantum is assumed to be of the same order as the momentum of the particle, so that $k_\perp/m$, k_z/m and p'_z/m are of the same order as p_z/m. For the quantity $p_n/m = (2nB/B_c)^{1/2}$ to be small for modest values of n, one also needs to make the weak-field approximation $B \ll B_c$.

With the foregoing approximations, the quantities defined in equation (6) reduce to

$$\lambda'_+ \approx \lambda_+ \approx \beta'_+ \approx \beta_+ \approx (2m)^{1/2} \qquad \lambda'_- \approx \frac{|p'_z|}{(2m)^{1/2}}$$
$$\lambda_- \approx \frac{|p_z|}{(2m)^{1/2}} \qquad \beta'_- \approx \frac{p_{n'}}{(2m)^{1/2}} \qquad \beta_- \approx \frac{p_n}{(2m)^{1/2}}. \tag{10}$$

These approximations allow one to order the coefficients of the J-functions in (6) in powers of the small quantities.

For a transition that involves a jump of $n - n' = j$, the leading terms in (6) are those with lower index $n' - n + 1 = -(j - 1)$. These terms appear only in the one- and two-components of $\mathbf{\Gamma}$. For a non-spin-flip transition, $s' = s$, these are the only terms that need to be retained. However, for $s = 1,\ s' = -1$, the coefficients of these terms are one order higher in the small quantities, and the three-component of $\mathbf{\Gamma}$ cannot be neglected.

The relevant approximation to (6) is different for transitions involving no spin flip ($s = s'$), those involving a spin flip ($s = 1 \rightarrow s' = -1$) and those involving a reverse spin flip ($s = -1 \rightarrow s' = 1$). There are three factors in the evaluation of $\mathbf{\Gamma}$ (cf (6)) and each of these needs to be evaluated for the four different combinations of spins for specified $j = n - n'$. The first of these is the multiplicative factor $b'^*_{n'} b_n = (\mathrm{i}\mathrm{e}^{\mathrm{i}\psi})^j/(2m)^2$, which is common to all cases. The second factor involves the combination of λs, which are approximated using (10). The signs $P',\ P$ ensure that in the approximations to λ'_- and λ_-, the modulus signs are removed from p'_z and p_z, respectively. There is a remaining overall sign that can involve P' or P, but which is of no physical relevance and is ignored in the following. Then one finds (for $j > 0$)

$$\begin{aligned}
\lambda'_{s'}\lambda_s + \lambda'_{-s'}\lambda_{-s} &\approx \tfrac{1}{2}(1 + s's)2m + \tfrac{1}{2}(1 - s's)(p'_z + p_z) \\
\lambda'_{s'}\lambda_s - \lambda'_{-s'}\lambda_{-s} &\approx s\left[\tfrac{1}{2}(1 + s's)2m + \tfrac{1}{2}(1 - s's)(p'_z - p_z)\right] \\
\lambda'_{s'}\lambda_{-s} + \lambda'_{-s'}\lambda_s &\approx \tfrac{1}{2}(1 + s's)(p'_z + p_z) + \tfrac{1}{2}(1 - s's)2m.
\end{aligned} \tag{11}$$

The other factor involves the J-functions. Retaining only the lowest order terms in $x^{1/2}$, $(B/2B_c)^{1/2}$ for each value of s, s', one finds

$$\beta'_{s'}\beta_s J^{n-1}_{n'-n} + s's\beta'_{-s'}\beta_{-s} J^n_{n'-n} \approx 2m\frac{(-)^j}{j!}\left(\frac{l!}{(l-j)!}\right)^{1/2} x^{j/2}$$
$$\times \begin{cases} 1 & \text{for} \quad s = s' \\ -j(B/2B_c)^{1/2}/(l - j + 1)^{1/2} & \text{for} \quad s = -s' = 1 \\ \mathrm{O}(B/B_c, x) & \text{for} \quad s = -s' = -1 \end{cases}$$
$$\beta'_{-s'}\beta_s J^{n-1}_{n'-n+1} \mp s's\beta'_{s'}\beta_{-s} J^n_{n'-n-1} \approx 2m\frac{(-)^{j-1}}{(j-1)!}\left(\frac{l!}{(l-j)!}\right)^{1/2} x^{(j-1)/2}$$
$$\times \begin{cases} (B/2B_c)^{1/2} & \text{for} \quad s = s' \\ 1/(l - j + 1)^{1/2} & \text{for} \quad s = -s' = 1 \\ \mathrm{O}(B/B_c, x) & \text{for} \quad s = -s' = -1. \end{cases} \tag{12}$$

for $\psi = 0$ and with $l = n - \frac{1}{2}(1 + s)$. In the entries for the reverse spin flip ($s = -1$, $s' = 1$) in the first equation in (12) the leading terms cancel.

There is a contribution to the vertex function (6) for $j = 0$, which may be approximated by

$$\boldsymbol{\Gamma}_{n,n,s,s'}(\boldsymbol{k}) = \tfrac{1}{2}(1 + s's)\,[(p'_z + p_z)/2m]\,(0, 0, 1). \tag{13}$$

However, transitions with $j = 0$ correspond to Cerenkov-like emission and not to gyromagnetic emission. Hence, the term (13) is of no direct interest for gyromagnetic emission.

Collecting terms, the relevant approximations to the vertex function (6) for gyromagnetic emission by a non-relativistic electron are

$$\begin{aligned}\boldsymbol{\Gamma}_{n,n-j,s,s'}(\boldsymbol{k}) = (-\mathrm{i})^j \Bigg[&\tfrac{1}{2}(1 + ss')\left(\frac{B}{2B_c}\right)^{1/2}\left(\frac{l!}{(l-j)!}\right)^{1/2}\frac{x^{(j-1)/2}}{(j-1)!}(1, \mathrm{i}, 0) \\ &+ \tfrac{1}{2}(1 + s)\tfrac{1}{2}(1 - s')\left(\frac{l!}{(l-j+1)!}\right)^{1/2}\frac{x^{(j-1)/2}}{(j-1)!}\frac{(k_z, \mathrm{i}k_z, -k_\perp)}{2m}\Bigg]\end{aligned} \tag{14}$$

with $(p'_z - p_z)/m = -k_z/m$. The two terms inside the square brackets correspond to the non-spin-flip and spin-flip transitions, respectively, and the reverse spin flip is neglected. For a spin-flip transition, assuming k_z is the same order as $k_\perp$, $\boldsymbol{\Gamma}$ is of order $x^{1/2}$ times that for a non-spin-flip transition with the same j.

In the approximation (14) the non-spin-flip transitions correspond to 2^j-electric multipole radiation with $j = n - n' = l - l'$, so that j corresponds to the change in the orbital quantum number. Specifically, the transition with $s' = s$ and $j = l - l' = 1$ corresponds to electric dipole radiation, the transition with $s' = s$ and $j = l - l' = 2$ corresponds to electric quadrupole radiation, and so on. The direct spin-flip transition $s = 1$, $s' = -1$ for $j = n - n' = 1$ involves no change in the orbital quantum number, $l' = l$, and this corresponds to magnetic dipole radiation.

Repeating the calculation for a positron rather than an electron, the result is equivalent to using the complex conjugate of the vertex function (14). This implies that emission by a positron differs from that due to an electron only in that the circular polarization of the emission has the opposite handedness.

3.3. Probability for cyclotron emission

The probability (5) for gyromagnetic emission in general reduces to the probability for cyclotron emission when the non-relativistic approximation is made, including the approximation (14) to the vertex function. It is convenient to choose two orthogonal elliptical polarizations, with axial ratios T and $-1/T$; complete information on the polarization is then contained in a single probability with an arbitrary T. This involves choosing the polarization vector to be

$$\boldsymbol{e} = \frac{T\boldsymbol{t} + \mathrm{i}\boldsymbol{a}}{(1 + T^2)^{1/2}} \tag{15}$$

where the direction of wave propagation, $\boldsymbol{\kappa} = (\sin\theta\cos\psi, \sin\theta\sin\psi, \cos\theta)$, and $\boldsymbol{t} = (\cos\theta\cos\psi, \cos\theta\sin\psi, -\sin\theta)$, $\boldsymbol{a} = (-\sin\psi, \cos\psi, 0)$ form an orthonormal triad. The probability for a non-spin-flip transition, $n \to n' = n - j$ with $s' = s$, is

$$\begin{aligned} w_{n,n-j,s,s}(\boldsymbol{k}) = &\frac{\mu_0 e^2(1 + T\cos\theta)^2}{2\omega(1 + T^2)}\frac{l!\sin^{2(j-1)}\theta}{2^j(l-j)![(j-1)!]^2} \\ &\times\left(\frac{B}{B_c}\right)^{2-j}\left(\frac{\omega}{m}\right)^{2(j-1)} 2\pi\delta(\varepsilon_n - \varepsilon'_{n-j} - \omega)\end{aligned} \tag{16}$$

where the weak dispersion approximation $\left(\omega^2 = k_\perp^2 + k_z^2\right)$ is made, and the non-relativistic approximation is not made to the δ-function. The probability for a transition $n \to n' = n - j$ with a spin flip, $s = 1 \to s' = -1$, is

$$w_{n,n-j,+,-}(\boldsymbol{k}) = \frac{\mu_0 e^2 (\cos\theta + T)^2}{2\omega(1+T^2)} \frac{l!\,\sin^{2(j-1)}\theta}{2^{j+1}(l-j+1)![(j-1)!]^2} \times \left(\frac{B}{B_c}\right)^{1-j} \left(\frac{\omega}{m}\right)^{2j} 2\pi\,\delta(\varepsilon_n - \varepsilon'_{n-j} - \omega). \tag{17}$$

The probabilities corresponding to (16), (17) for a positron may be obtained by replacing the polarization vector by its complex conjugate, which is equivalent to $T \to -T$ in (16) and (17).

3.4. Non-spin-flip transitions

The rate of transitions per unit time is found by integrating the probability over $\mathrm{d}^3\boldsymbol{k}/(2\pi)^3$, and summing over the two orthogonal polarizations if one has no interest in the polarization of the emission. The non-spin-flip ($s' = s$) transition rate for cyclotron emission *in vacuo* is given by

$$R_{n,n',s,s} = \frac{\mu_0 e^2}{2} \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} \frac{|\boldsymbol{\kappa} \times \Gamma_{n,n',s,s}(\boldsymbol{k})|^2}{\omega} 2\pi\,\delta(\varepsilon_n - \varepsilon'_{n'} - \omega). \tag{18}$$

The rates for emission of photons polarized perpendicular ($T = 0$) and parallel ($T = \infty$) may be evaluated separately: their sum gives the total rate $R_{n,n'}$, and their difference divided by their sum, denoted $r_{n,n'}$, gives information on the degree of linear polarization of the emitted photons.

For non-spin-flip transitions, $R^{\mathrm{nsf}}_{l,l-j} = R_{n,n-j,s,s}$, with $n = l + \frac{1}{2}(1+s)$, the rate reduces to

$$R^{\mathrm{nsf}}_{l,l-j} = \alpha m \frac{l!}{(l-j)!} \frac{2^{j+1}(j+1)!}{(j-1)!(2j+1)!} \left(\frac{\omega}{m}\right)^{2j-1} \left(\frac{B}{B_c}\right)^{2-j} \tag{19}$$

with $\omega = \varepsilon_n - \varepsilon_{n-j}$. To lowest order in B/B_c, $\omega = j\Omega_e = jm(B/B_c)$ is the jth harmonic of the cyclotron frequency, and then (19) implies $R^{\mathrm{nsf}}_{l,l-j} \propto (B/B_c)^{j+1}$. The polarization for emission at a given angle θ is right-hand elliptical with major axis along $\boldsymbol{a}$ and axial ratio $T = 1/|\cos\theta|$. The rate of emission can be evaluated separately for the orthogonal linear polarizations. The integrated degree of linear polarization

$$r^{\mathrm{nsf}}_{l,l-j} = \frac{j}{j+1} \tag{20}$$

implies a preference for emission of a photon polarized along $\boldsymbol{a}$ rather than $\boldsymbol{t}$.

3.5. Spin-flip transitions

For spin-flip transitions, $R^{\mathrm{sf}}_{l,l-j} = R_{n,n-j,+,-}$, with $l = n-1$, $l' = n' = n - j = l - j + 1$, one finds

$$R^{\mathrm{sf}}_{l,l-j} = \alpha m \left(\frac{\omega}{m}\right)^{2j+1} \left(\frac{B}{B_c}\right)^{-j+1} \frac{l!}{(l-j+1)!} \frac{2^j j(j+1)}{(2j+1)!}. \tag{21}$$

The approximation $\omega = jm(B/B_c)$ in (28) implies $R^{\mathrm{sf}}_{l,l-j} \propto (B/B_c)^{j+2}$, one power in B/B_c higher than for the non-spin-flip transition (19). The polarization for emission at a given angle is right-hand elliptical with major axis along $\boldsymbol{t}$ with axial ratio $T = 1/|\cos\theta|$. The counterpart of (20)

$$r^{\mathrm{sf}}_{l,l-j} = -\frac{j}{j+1} \tag{22}$$

implies a preference for emission of a photon polarized along $\boldsymbol{t}$ rather than $\boldsymbol{a}$.

3.6. Reverse spin-flip transitions

For reverse spin-flip transitions, $s = -1 \rightarrow s' = 1$, the leading terms in the non-relativistic approximation to the vertex function cancel. It is convenient to use the exact expressions for the J-functions, and to simplify the result using recursion relations that they satisfy. The recursion relation used is

$$l^{1/2} J_{-j}^{l-1}(x) - (l-j)^{1/2} J_{-j}^{l}(x) = -x^{1/2} J_{-j-1}^{l}(x). \tag{23}$$

Collecting terms, the vertex function for a reverse spin flip, $\boldsymbol{\Gamma}_{l,l-j}^{\mathrm{rsf}}(\boldsymbol{k}) = \boldsymbol{\Gamma}_{n,n-j,-1,1}(\boldsymbol{k})$ with $l = n$, reduces to

$$\boldsymbol{\Gamma}_{l,l-j}^{\mathrm{rsf}}(\boldsymbol{k}) = (-\mathrm{i})^{j} \left[J_{-j-1}^{l}(x) \frac{(k_z, -\mathrm{i}k_z, -k_\perp)}{2m} - \frac{k_z}{2m} \frac{B}{2B_c} [l(l-j)]^{1/2} J_{-j+1}^{l-1}(x)(1, \mathrm{i}, 0) \right]. \tag{24}$$

Then comparing (24) and (14) and making the approximation (9) to the J-functions with $x, k_z/m$ and $k_\perp/m$ all of order B/B_c, one finds that $\boldsymbol{\Gamma}$ for a reverse spin-flip transition is of order $(B/B_c)^{3/2}$ smaller than that for a non-spin-flip transition, and of order B/B_c smaller than that a spin-flip transition with the same j. The rate of transition, being proportional to $|\Gamma|^2$, is smaller for a reverse spin-flip transition of the order $(B/B_c)^3$ than for a non-spin-flip transition, and of the order $(B/B_c)^2$ than for a spin-flip transition with the same j. Thus, a reverse spin-flip transition at the fundamental, $j = 1$, is of the same order as a non-spin-flip transition with $j = 4$ or of a spin-flip transition with $j = 3$. In the context of cyclotron emission theory, reverse spin-flip transitions are negligible whenever emission higher than the third harmonic is negligible.

4. Transitions between helicity eigenstates

The choice of the spin operator as the magnetic moment operator is the only acceptable one when discussing spin flips (Sokolov and Ternov 1968, 1986, Herold *et al* 1982). To emphasize this point, in this section we discuss the implications of choosing an alternative spin operator, specifically the helicity.

4.1. Transitions between helicity eigenstates

The helicity operator is written down in appendix C. Its eigenvalues are written as $\sigma P h_n$, with $\sigma = \pm 1$, $P = p_z/|p_z|$, $h_n = \left(p_n^2 + p_z^2\right)^{1/2}$. The sign P is needed so that the ground state corresponds to $\sigma = -1$. One might identify a spin flip as $\sigma \rightarrow -\sigma$ in this case. However, for the helicity states there are two complications in defining what is meant by a spin flip.

One complication is related to the interpretation of the case $P'P = -1$. As shown in appendix C, choosing the ground state to be $\sigma = -1$ requires the appearance of the sign P in the eigenvalue and the eigenfunctions. Suppose an electron with $p_z > 0$ emits a photon with $k_z > p_z$ to give $p_z' = p_z - k_z < 0$. The helicity then changes sign due to the change in sign of the momentum, and this is not what one conventionally thinks of as a spin flip. This complication is ignored in the following discussion by assuming $P'P = 1$.

The other complication has wider implications: the radiative correction that leads to the familiar correction $\left(\frac{1}{2}\right)(g-2) = \alpha/2\pi$ to the g-factor in the anomalous magnetic moment of the electron (Schwinger 1949) causes all spin operators other than the magnetic moment to precess at a frequency $\Omega_e \alpha/2\pi$, $\Omega_e = eB/m$ (Sokolov and Ternov 1968, 1986, Parle 1987). The precession causes the expectation value of the helicity operator to vary periodically at this frequency, and such changes could be misinterpreted as a spin flip. In fact, the helicity eigenstates are well defined only on times much shorter than the precession period, and it is meaningful to discuss radiative transitions between eigenstates only on such short times. The concept of a spin flip is meaningful only if the rate of spin-flip transitions exceeds the precession rate. Consider two alternative criteria: that the slowest of the non-spin-flip transition rates, $n = 1 \to n' = 0$, exceeds the precession rate, and that the spin-flip transition rate exceeds the precession rate. These reduce to

$$\frac{R^{\rm nsf}_{1,0}}{\Omega_e\alpha/2\pi} = \frac{8\pi}{3}\frac{B}{B_c} > 1 \qquad \frac{R^{\rm sf}_{l,l-1}}{\Omega_e\alpha/2\pi} = \frac{4\pi}{3}\left(\frac{B}{B_c}\right)^2 > 1 \tag{25}$$

where (19) with $l = 1, j = 1, \omega = \Omega_e$ and (28) with $j = 1, \omega = \Omega_e$ are used, respectively. The former of these is not consistent with the condition $2B/B_c \ll 1$ required for the non-relativistic approximation to apply, and the latter fails to satisfy this criterion even more strongly. We conclude that it is inconsistent to discuss spin-flip transitions due to cyclotron emission for any spin operator apart for the magnetic moment operator.

4.2. Transition rates

Nevertheless, for the purpose of discussion, we calculate the transition rates due to cyclotron emission *in vacuo* for the four possible helicity transitions, $\sigma = \sigma' = 1, \sigma = \sigma' = -1, \sigma = -\sigma' = 1, \sigma = -\sigma' = -1$. The calculation, outlined in appendix C, gives $\big($with $n = l + \frac{1}{2}(1+\sigma)$ for the helicity states$\big)$

$$\begin{aligned}
R^{\rm hel}_{n,n-j,+,+} &= \bar{A}_{n,n-j}(h_n + h'_{n'})^2\cos^2\left(\tfrac{1}{2}\alpha_n\right)\sin^2\left(\tfrac{1}{2}\alpha'_{n'}\right)\\
R^{\rm hel}_{n,n-j,-,-} &= \bar{A}_{n,n-j}(h_n + h'_{n'})^2\sin^2\left(\tfrac{1}{2}\alpha_n\right)\cos^2\left(\tfrac{1}{2}\alpha'_{n'}\right)\\
R^{\rm hel}_{n,n-j,+,-} &= \bar{A}_{n,n-j}(h_n - h'_{n'})^2\cos^2\left(\tfrac{1}{2}\alpha_n\right)\cos^2\left(\tfrac{1}{2}\alpha'_{n'}\right)\\
R^{\rm hel}_{n,n-j,-,+} &= \bar{A}_{n,n-j}(h_n - h'_{n'})^2\sin^2\left(\tfrac{1}{2}\alpha_n\right)\sin^2\left(\tfrac{1}{2}\alpha'_{n'}\right)
\end{aligned} \tag{26}$$

with $h_n = \left(p_n^2 + p_z^2\right)^{1/2}$, $h'_{n'} = \left(p_{n'}^2 + p_z'^2\right)^{1/2}$, and with

$$\bar{A}_{n,n-j} = \frac{\alpha m}{2eB}\frac{(n-1)!}{(n-j)!}\frac{2^{j+1}(j+1)!}{(j-1)!(2j+1)!}\left(\frac{\omega}{m}\right)^{2j-1}\left(\frac{B}{B_c}\right)^{2-j} \tag{27}$$

where the angles $\alpha_n, \alpha'_{n'}$ are defined by analogy with the classical pitch angle, by writing $p_n = h_n\sin\alpha_n,\ p_z = h_n\cos\alpha_n,\ p_{n'} = h'_{n'}\sin\alpha'_{n'}$, and $p'_z = h'_{n'}\cos\alpha'_{n'}$.

The transition rates for the helicity and magnetic moment states can be compared by averaging over the initial spins and summing over the final spins. For the helicity states (26), (27) imply

$$\sum_{\sigma,\sigma'} R^{\rm hel}_{n,n-j,\sigma,\sigma'} = \bar{A}_{n,n-j}\left[p_n^2 + p_{n'}^2 + (p_z - p'_z)^2\right] \approx \bar{A}_{n,n-j}(2n-j)(2eB). \tag{28}$$

In the final step the term $(p_z - p'_z)^2 = k_z^2 \approx (jeB/m)^2\cos^2\theta$ is neglected in comparison with $p_n^2 + p_{n'}^2 = (2n-j)(2eB)$ due to it being one order higher in B/B_c. Performing the sum over the rate of non-spin-flip transitions for the magnetic moment states, that is over (19) for $l = n$ and for $l = n-1$, one obtains the same result as (28) to lowest order in B/B_c. It follows that all the rates (26) are of the same order as the non-spin-flip transitions for the magnetic moment states.

An interpretation of the foregoing result involves regarding the helicity states as mixtures of the magnetic moment states. The coefficients relating the two are $\cos\left(\frac{1}{2}\alpha_n\right), \sin\left(\frac{1}{2}\alpha_n\right)$. A change $n \to n'$ implies a change in the coefficients $\cos\left(\frac{1}{2}\alpha_n\right), \sin\left(\frac{1}{2}\alpha_n\right) \to \cos\left(\frac{1}{2}\alpha'_{n'}\right)$,

$\sin\left(\frac{1}{2}\alpha'_{n'}\right)$. Suppose the initial state is pure $\sigma = 1$. This corresponds to a mixture of the $s = 1$, -1 states with coefficients $\cos\left(\frac{1}{2}\alpha_n\right)$, $\sin\left(\frac{1}{2}\alpha_n\right)$. A non-spin-flip transition (preserving s) does not change this mixture. However, for the new n' this mixture does not correspond to the pure state $\sigma' = 1$ because the coefficients are not the required $\cos\left(\frac{1}{2}\alpha'_{n'}\right)$, $\sin\left(\frac{1}{2}\alpha'_{n'}\right)$. Thus $s' = s$ implies $\sigma' \neq \sigma$ and vice versa. The definition of a spin-flip transition is dependent on the choice of spin operator, and the magnetic moment is the appropriate choice when discussing gyromagnetic emission.

5. Effect of spin-flip transitions

In this section we discuss the effect of spin flips in cyclotron emission on the steady-state distribution of electrons in the Landau levels when there is a constant source of electrons at high levels. The question addressed is whether electrons with $s = 1$ preferentially reach the true ground state, $l = 0$, $s = -1$, (a) by a spin-flip transition at $l > 0$, or (b) by relaxing to the ground state, $l = 0$, $s = 1$, for $s = 1$ before making the spin-flip transition.

5.1. Kinetic equations

Let N_n^s denote the occupation number of the state n, s integrated over the parallel momentum, p_z. The evolution of these occupation numbers due to cyclotron emission is described by the kinetic equations

$$\begin{aligned}\frac{\mathrm{d}N_n^+}{\mathrm{d}t} &= \sum_{j=1}[R_{n+j,n,+,+}N_{n+j}^+ - (R_{n,n-j,+,+} + R_{n,n-j,+,-})N_n^+]\\ \frac{\mathrm{d}N_n^-}{\mathrm{d}t} &= \sum_{j=1}[R_{n+j,n,-,-}N_{n+j}^- - (R_{n,n-j,-,-})N_n^- + R_{n+j,n,+,-}N_{n+j}^+]\end{aligned} \tag{29}$$

with the transition rates given by (19) and (28).

To lowest order in B/B_c only the non-spin-flip transitions with $j = 1$ are retained in (29). In this case the only rate retained in (29) is

$$R_{l,l-1,s,s} = lR_0 \qquad R_0 = \frac{4\alpha m}{3}\left(\frac{B}{B_c}\right)^2. \tag{30}$$

The steady-state solution of (29) is independent of s, and it is convenient to write $N_n^s = N_l$. Assuming an injection at a rate $\dot{N}_L$ at some level $l = L$, the solution is

$$N_l = \frac{\dot{N}_L}{lR_0}. \tag{31}$$

The number in the lowest state, $l = 0$, for each spin increases at the rate $\dot{N}_L$ that the electrons are injected at $l = L$.

5.2. Inclusion of spin flips

The inclusion of spin flips allows electrons to jump from the states l, $s = 1$ to l, $s = -1$ at the rate

$$R^{\mathrm{sf}} = R_0\frac{B}{2B_c}. \tag{32}$$

All electrons end up in the ground state with $l = 0$, $s = -1$. There is no change in the solution (31) on electrons initially with $s = -1$. The question of interest concerns electrons initially with $s = 1$: do the electrons tend to make the jump $s = 1 \to -1$ in an excited state $l > 0$, or do they tend to collect in the state $l = 0$, $s = 1$ before making the spin-flip transition? The rate of spin-flip transitions at a given l is independent of l according to (32), and hence it is proportional to N^+_{l+1}. In a perturbation approach, the effect of the spin-flip transitions on the solution for $s = 1$ is neglected, so that for $l > 0$ (31) implies $N^+_{l+1} = \dot{N}_L / l R_0$. The lowest energy state, $l = 0$, reaches a steady state with

$$N^+_1 = \frac{\dot{N}_L}{R^{\mathrm{sf}}} = \frac{\dot{N}_L}{R_0} \left(\frac{B}{2B_c} \right)^{-1} \tag{33}$$

where (32) is used. It follows that the rate of transitions $s = 1 \to -1$ is

$$\frac{\mathrm{d}N^+_{l+1}}{\mathrm{d}t} = \begin{cases} \dfrac{\dot{N}_L}{l} \dfrac{B}{2B_c} & \text{for} \quad l > 0 \\ \dot{N}_L & \text{for} \quad l = 0. \end{cases} \tag{34}$$

One concludes that there is a preference for the electrons to relax to the state $l = 0$, $s = 1$ before making the spin-flip transition, but that the rate of spin-flip transitions at higher l is not necessarily negligible. For example, suppose one has $B/2B_c = 0.1$, so that the non-relativistic approximation is valid only for $l \ll 1/(B/2B_c) = 10$. Assuming $L = 1/(B/2B_c) = 10$, the sum of the rates (34) for $0 < l \leqslant L$ is about 30% of the rate for $l = 0$. This branching ratio decreases with decreasing $B/2B_c$ for $L = 1/(B/2B_c)$, implying that the preference for relaxation to $l = 0$, $s = 1$ before the spin-flip transition increases with decreasing B. With $L \sim 1/(B/B_c)$ the branching ratio of spin-flip transitions at $l > 0$ compared with those at $l = 0$ decreases $\sim(\ln L)/L$ for $B/B_c \to 0$.

6. Discussion and conclusions

The main point emphasized in this paper is that a correct treatment of spin dependence in the quantum theory of cyclotron emission requires (a) the use of the non-relativistic limit of Dirac theory (rather than use of the Schrödinger–Pauli theory) and (b) the choice of the magnetic moment as the spin operator. The use of intrinsically relativistic theory to treat a seemingly non-relativistic problem is due to spin–orbit coupling being important even in the non-relativistic limit, and spin–orbit coupling is an intrinsically relativistic effect.

The magnetic-moment operator is the only acceptable choice of spin operator when treating gyromagnetic emission. An acceptable spin operator must commute with the (Dirac) Hamiltonian, and all other operators that satisfy this requirement precess at a rate $(\alpha/2\pi)\Omega_e$ due to the radiative correction to the magnetic moment (Schwinger 1949, Sokolov and Ternov 1968, 1986, Parle 1987). According to (25), the transition rates for cyclotron emission are typically less than the precession rate, implying that a spin-flip transition for other spin operators cannot be defined meaningfully. Nevertheless, for illustrative purposes, cyclotron emission is treated for the helicity states, $\sigma = \pm$, and it is argued in section 4 that transitions in which σ changes sign should not be interpreted as a true spin flip.

In section 3 we show that the transition rate for a reverse spin-flip is of order $(B/B_c)^3$ higher than that for a non-spin-flip transition, and of order $(B/B_c)^2$ higher than that for a direct spin-flip transition. This implies that reverse spin-flips due to cyclotron emission are effectively forbidden. In an earlier treatment (Melrose and Zheleznyakov 1981) the Johnson–Lippmann (Johnson and Lippmann 1949) states were chosen, and this precludes a correct treatment of a reverse spin flip. The Johnson–Lippmann wavefunctions reduce to the magnetic moment eigenstates for an electron, but not for a positron, in a non-relativistic approximation, but higher order terms need to be retained to treat the reverse spin flip correctly. The correct rate for reverse spin-flip transitions requires that one choose the correct spin eigenstates.

Sokolov and Ternov (1968, 1986) showed that as a result of synchrotron emission, electrons tend to become 96% polarized in the state $s = -1$, and we discuss the possible counterpart of this result for cyclotron emission. The absence of reverse spin flips implies that the only spin change that can result from cyclotron emission is $s = 1 \to -1$. For an electron in an excited state $l = L \sim (B/2B_c)^{-1}$ with $s = 1$, the results of section 5 imply that there is a preference for the electron to jump sequentially ($l \to l - 1$) down to the state $l = 0$, $s = 1$ before making the spin-flip transition to the ground state $l = 0$, $s = -1$. Thus, as in the synchrotron case, cyclotron emission favours electrons collecting in the state $s = -1$. For weak fields, $B \ll B_c$, the relative probability of the spin flip occurring from an excited state, $l > 0$, rather than for $l = 0$, approaches zero for $B \to 0$ as $(\ln L)/L,\ L \sim (B/2B_c)^{-1}$.

We conclude with a few remarks on applications of the results derived here. The quantum theory of cyclotron emission is needed to complement the long-established quantum theory of synchrotron emission, which was motivated by laboratory applications (Sokolov and Ternov 1968). As a result of synchrotron emission an electron loses its perpendicular energy, and enters the cyclotron regime when the perpendicular energy becomes non-relativistic. In this sense a discussion of the cyclotron regime is a necessary addition to the synchrotron regime. The results derived here are also needed to discuss spin-dependent effects in cyclotron absorption and on wave dispersion in dense, strongly-magnetized electron gases. Absorption may be related to emission by appealing to detailed balance using the probability (5). Also, a treatment of dispersion involves the same approximations to the vertex functions as derived here. A particular application of interest to us is of pulsars, which are neutron stars with superstrong magnetic fields. The vast bulk of ‘ordinary’ pulsars have surface field $B/B_c \sim 0.1$, with ‘millisecond pulsars’ and ‘magnetars’ having much weaker and stronger fields, respectively (e.g., Taylor *et al* 1993, Thompson and Duncan 1996). A pulsar magnetosphere is populated by electron–positron pairs produced through one-photon pair creation. Recently it has been shown (Weise and Melrose 2001) that for $B/B_c \gtrsim 0.1$ the resulting pairs are in low Landau levels $n \ll (2B/B_c)^{-1}$, confirming earlier suggestions that this might be the case (Daugherty and Harding 1983). As a consequence, the gyromagnetic emission by these particles is in the cyclotron regime, rather than the synchrotron regime as has usually been assumed (e.g., Daugherty and Harding 1996). These various applications will be discussed elsewhere.

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Appendix A. Wavefunctions

Dirac’s equation for an electron in a magnetic field has energy eigenvalues $\epsilon\varepsilon_n$, with $\epsilon = \pm 1$, $\varepsilon_n = \left(m^2 + p_z^2 + p_n^2\right)^{1/2}$, $p_n = (2neB)^{1/2}$. It is convenient to write the wavefunction in the form

$$\Psi_q^{\epsilon}(t, \boldsymbol{x}) = \mathrm{e}^{-\mathrm{i}\epsilon\varepsilon_n t}\Psi_q^{\epsilon}(\boldsymbol{x}) \tag{A.1}$$

where q denotes the eigenvalues, which include n, p_z, a gauge-dependent quantum number, g, and the spin quantum number, $s = \pm 1$. One is free to choose a wavefunction of the factorized form (Ritus 1970, 1972, Parle 1987)

$$\Psi_q^\epsilon(\boldsymbol{x}) = \mathcal{V}_g^\epsilon(\boldsymbol{x}, n, \epsilon p_z)\varphi_s^\epsilon(n, \epsilon p_z)\mathrm{e}^{\mathrm{i}p_z z} \qquad \varphi_s^\epsilon(n, \epsilon p_z) = \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} \tag{A.2}$$

where $\mathcal{V}_g^\epsilon(\boldsymbol{x}, n, \epsilon p_z)$ is a square matrix. In the Landau gauge, $\boldsymbol{A} = (0, Bx, 0)$, y is ignorable, and with the wavefunction chosen to be proportional to $\exp(\mathrm{i}\epsilon p_y y)$, the identification $g \to p_y$ is made in (A.2). In the standard representation of the Dirac algebra, one then has

$$\mathcal{V}_g^\epsilon(\boldsymbol{x}, n, \epsilon p_z) = \mathrm{e}^{\mathrm{i}\epsilon p_y y} \begin{pmatrix} v_{n-1}(\xi) & 0 & 0 & 0 \\ 0 & v_n(\xi) & 0 & 0 \\ 0 & 0 & v_{n-1}(\xi) & 0 \\ 0 & 0 & 0 & v_n(\xi) \end{pmatrix} \tag{A.3}$$

which involves the simple harmonic oscillator wavefunctions

$$v_n(\xi) = \frac{1}{(\sqrt{\pi}2^n n!)^{1/2}} H_n(\xi)\mathrm{e}^{-\xi^2/2} \qquad \xi = (eB)^{1/2}\left(x + \frac{\epsilon p_y}{eB}\right) \tag{A.4}$$

where H_n is a Hermite polynomial.

The $C_1, \ldots, C_4$ are eigenfunctions of the Hamiltonian, which implies

$$\begin{pmatrix} m & 0 & \epsilon p_z & -\mathrm{i}p_n \\ 0 & m & \mathrm{i}p_n & -\epsilon p_z \\ \epsilon p_z & -\mathrm{i}p_n & -m & 0 \\ \mathrm{i}p_n & -\epsilon p_z & 0 & -m \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \epsilon\varepsilon_n \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix}. \tag{A.5}$$

The solutions of (A.5) are doubly degenerate, corresponding to the positive and negative energy eigenvalues $\epsilon\varepsilon_n$, and one requires that the $C_1, \ldots, C_4$ also be simultaneous eigenvalues of a spin operator in order to find the solutions (4).

Appendix B. The vertex function

The vertex function is defined by

$$\left[\gamma_{q'q}^{\epsilon'\epsilon}(\boldsymbol{k})\right]^\mu = \int \mathrm{d}^3\boldsymbol{x}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\, \bar{\Psi}_{q'}^{\epsilon'}(\boldsymbol{x})\gamma^\mu \Psi_q^\epsilon(\boldsymbol{x}) \tag{A.6}$$

where γ^μ, $\mu = 0, \ldots, 3$ are the Dirac matrices, and the overline denotes the Dirac adjoint. The wavevector is written in the form $\boldsymbol{k} = (k_\perp \cos\psi, k_\perp \sin\psi, k_z)$. The vertex function (A.6) factorizes:

$$\left[\gamma_{q'q}^{\epsilon'\epsilon}(\boldsymbol{k})\right]^\mu = d_{q'q}^{\epsilon'\epsilon}(\boldsymbol{k}) \left[\Gamma_{q'q}^{\epsilon'\epsilon}(\boldsymbol{k})\right]^\mu \tag{A.7}$$

with $d_{q'q}^{\epsilon'\epsilon}(\boldsymbol{k})$ containing all the gauge-dependent parts. For the Landau gauge one has

$$d_{q'q}^{\epsilon'\epsilon}(\boldsymbol{k}) = \frac{\mathrm{e}^{\mathrm{i}k_x(\epsilon p_y + \epsilon' p_y')/2eB}}{V(eB)^{1/2}} 2\pi\delta(\epsilon p_y - \epsilon' p'_y - k_y) 2\pi\delta(\epsilon p_z - \epsilon' p'_z - k_z) \tag{A.8}$$

where conservation of the y-component of momentum, $\epsilon' p'_y = \epsilon p_y - k_y$, is used implicitly in the factor $\mathrm{e}^{-\mathrm{i}\psi} = [k_x - \mathrm{i}(\epsilon p_y - \epsilon' p'_y)]/k_\perp$. This part of the vertex function does not appear explicitly in the theory of gyromagnetic emission, or for any process that does not depend on the position of the gyrocentre of the electron.

We refer to the quantity $\mathbf{\Gamma}$ as the vertex function; it is the space part of the 4-vector introduced in (A.7), and is given by

$$\left[\Gamma^{\epsilon'\epsilon}_{q'q}(\boldsymbol{k})\right]^{\mu} = V\bar{\varphi}^{\epsilon'}_{s'}(n', \epsilon' p'_z)\, G^{\mu}(n', n, \boldsymbol{k})\, \varphi^{\epsilon}_{s}(n, \epsilon p_z). \tag{A.9}$$

with the ||-components ($\mu = 0, z$) and $\perp$-components ($\mu = x, y$) given by

$$G^{\mu}_{\parallel}(n', n, \boldsymbol{k}) = \gamma^{\mu}_{\parallel}\, \mathcal{J}_{\parallel}(n', n, \boldsymbol{k}) \qquad G^{\mu}_{\perp}(n', n, \boldsymbol{k}) = \gamma^{\mu}_{\perp}\mathcal{J}_{\perp}(n', n, \boldsymbol{k}) \tag{A.10}$$

$$\mathcal{J}_{\parallel} = (-\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\psi})^{n'-n} \begin{pmatrix} J^{n-1}_{n'-n} & 0 & 0 & 0 \\ 0 & J^{n}_{n'-n} & 0 & 0 \\ 0 & 0 & J^{n-1}_{n'-n} & 0 \\ 0 & 0 & 0 & J^{n}_{n'-n} \end{pmatrix} \tag{A.11}$$

$$\mathcal{J}_{\perp} = (-\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\psi})^{n'-n} \begin{pmatrix} -\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\psi} J^{n-1}_{n'-n+1} & 0 & 0 & 0 \\ 0 & \mathrm{i}\,\mathrm{e}^{\mathrm{i}\psi} J^{n}_{n'-n-1} & 0 & 0 \\ 0 & 0 & -\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\psi} J^{n-1}_{n'-n+1} & 0 \\ 0 & 0 & 0 & \mathrm{i}\,\mathrm{e}^{\mathrm{i}\psi} J^{n}_{n'-n-1} \end{pmatrix} \tag{A.12}$$

with the J-functions defined by (7) with argument $k_{\perp}^2/2eB$. The form (A.9) gives the explicit forms (6) and (A.15) for the magnetic-moment and helicity eigenstates, respectively.

Appendix C. Helicity states

The helicity operator in the Dirac theory in the presence of a magnetic field with vector potential $\boldsymbol{A}$ is

$$\boldsymbol{h} = \boldsymbol{\sigma} \cdot (\hat{\boldsymbol{p}} + e\boldsymbol{A}) \tag{A.13}$$

with the same notation as equation (1). The eigenvalues are $\sigma P h_n$, with $P = p_z/|p_z|$ and $h_n = \left(p_n^2 + p_z^2\right)^{1/2}$.

Simultaneous eigenstates of the helicity operator and the Hamiltonian in the standard representation of the Dirac algebra may be written in the same form as for the magnetic moment states. Equation (A.5) is replaced by

$$\begin{pmatrix} \epsilon p_z & -\mathrm{i}p_n & 0 & 0 \\ \mathrm{i}p_n & -\epsilon p_z & 0 & 0 \\ 0 & 0 & \epsilon p_z & -\mathrm{i}p_n \\ 0 & 0 & \mathrm{i}p_n & -\epsilon p_z \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \sigma\epsilon P h_n \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix}. \tag{A.14}$$

The simultaneous eigenfunctions are

$$\varphi^{\epsilon}_{\sigma}(n, \epsilon p_z) = \frac{1}{(2h_n 2\varepsilon_n V)^{1/2}} \begin{pmatrix} \Lambda_+\, g_{\sigma} \\ \mathrm{i}\sigma\epsilon P\, \Lambda_+\, g_{-\sigma} \\ \sigma P\, \Lambda_-\, g_{\sigma} \\ \mathrm{i}\epsilon\, \Lambda_-\, g_{-\sigma} \end{pmatrix}$$

$$\Lambda_{\pm} = (\varepsilon_n \pm \epsilon m)^{1/2} \qquad g_{\sigma} = (h_n + \sigma|p_z|)^{1/2}. \tag{A.15}$$

The appearance of the sign P is required so that the ground state ($n = 0$, $h_0 = |p_z|$) corresponds to $\sigma = -1$. For these helicity eigenstates, the vertex function is

$$\Gamma^{\mu} = a'^{*}_{n'} a_n \left([\Lambda'_+\Lambda_+ + S\Lambda'_-\Lambda_-]\left[g'_{\sigma'}g_{\sigma} J^{n-1}_{n'-n} + \epsilon'\epsilon S g'_{-\sigma'}g_{-\sigma} J^{n}_{n'-n}\right],\right.$$
$$- \epsilon' S[\Lambda'_+\Lambda_- + S\Lambda'_-\Lambda_+]\left[g'_{-\sigma'}g_{\sigma}\mathrm{e}^{-\mathrm{i}\psi} J^{n-1}_{n'-n+1} + \epsilon'\epsilon S g'_{\sigma'}g_{-\sigma}\mathrm{e}^{\mathrm{i}\psi} J^{n}_{n'-n-1}\right],$$

$$- \mathrm{i}\epsilon' S[\Lambda'_+\Lambda_- + S\Lambda'_-\Lambda_+] \left[g'_{-\sigma'} g_\sigma \mathrm{e}^{-\mathrm{i}\psi} J^{n-1}_{n'-n+1} - \epsilon'\epsilon S g'_{\sigma'} g_{-\sigma} \mathrm{e}^{\mathrm{i}\psi} J^{n}_{n'-n-1}\right],$$
$$\sigma P[\Lambda'_+\Lambda_- + S\Lambda'_-\Lambda_+] \left[g'_{\sigma'} g_\sigma J^{n-1}_{n'-n} - \epsilon'\epsilon S g'_{-\sigma'} g_{-\sigma} J^{n}_{n'-n}\right]\Big)$$

$$a_n = \frac{(\mathrm{i}\mathrm{e}^{\mathrm{i}\psi})^n}{(2h_n 2\varepsilon_n)^{1/2}} \qquad \Lambda_\pm = (\varepsilon_n \pm \epsilon m)^{1/2} \qquad S = P'P\sigma'\sigma \tag{A.16}$$

with $a'_{n'}$, $\Lambda'_\pm$, $g'_{\sigma'}$ defined in an analogous way in terms of primed quantities. The non-relativistic approximation to (A.16) involves making the approximation (9) to the J-functions, together with the following approximations, for an electron $\epsilon = \epsilon' = 1$:

$$\Lambda_+ \approx (2m)^{1/2} \qquad \Lambda_- \approx \frac{h_n}{(2m)^{1/2}} \qquad \Lambda'_+ \approx (2m)^{1/2} \qquad \Lambda'_- \approx \frac{h'_{n'}}{(2m)^{1/2}}. \tag{A.17}$$

One can set $PP' = 1$ except for $k_z > p_z > 0$ or $k_z < p_z < 0$, which need to be treated separately due to a 'spin flip' then arising from a purely kinematic change, as discussed in section 4. Assuming this special case can be treated separately, one can set $PP' = 1$. For $PP' = 1$, the non-relativistic approximation gives

$$\Lambda'_+\Lambda_- + \Lambda'_-\Lambda_+ = h_n + h'_{n'} \qquad \Lambda'_-\Lambda_+ - \Lambda'_+\Lambda_- = h'_{n'} - h_n. \tag{A.18}$$

It can be helpful to introduce the quantum counterpart, α, of the pitch angle by writing

$$p_n = h_n \sin\alpha_n \qquad p_z = h_n \cos\alpha_n \tag{A.19}$$

and similarly $p_{n'} = h'_{n'} \sin\alpha'_{n'}$ and $p'_z = h'_{n'} \cos\alpha'_{n'}$. Then for $p_z > 0$ and $PP' = 1$ one has

$$g_+ = (2h_n)^{1/2} \cos\left(\tfrac{1}{2}\alpha_n\right) \qquad g_- = (2h_n)^{1/2} \sin\left(\tfrac{1}{2}\alpha_n\right) \tag{A.20}$$

and similarly $g'_+ = (2h'_{n'})^{1/2} \cos\left(\tfrac{1}{2}\alpha'_{n'}\right)$ $\qquad g'_- = (2h'_{n'})^{1/2} \sin\left(\tfrac{1}{2}\alpha'_{n'}\right)$.

The lowest order terms in the expansion of the J-functions give, for $\sigma = \sigma' = 1, \sigma = \sigma' = -1, \sigma = -\sigma' = 1$ and $\sigma = -\sigma' = -1$, and for $\psi = 0$ and $n = l + \frac{1}{2}(1+\sigma)$

$$\Gamma_{n,n-j,+,+} = -\frac{\mathrm{i}^j(h_n + h'_{n'})}{2m} \cos\left(\tfrac{1}{2}\alpha_n\right) \sin\left(\tfrac{1}{2}\alpha'_{n'}\right) \; J^{l}_{-(j-1)}(1, \mathrm{i}, 0)$$
$$\Gamma_{n,n-j,-,-} = -\frac{\mathrm{i}^j(h_n + h'_{n'})}{2m} \sin\left(\tfrac{1}{2}\alpha_n\right) \cos\left(\tfrac{1}{2}\alpha'_{n'}\right) \; J^{l-1}_{-(j-1)}(1, \mathrm{i}, 0)$$
$$\Gamma_{n,n-j,+,-} = \frac{\mathrm{i}^j(h_n - h'_{n'})}{2m} \cos\left(\tfrac{1}{2}\alpha_n\right) \cos\left(\tfrac{1}{2}\alpha'_{n'}\right) \; J^{l}_{-(j-1)}(1, \mathrm{i}, 0)$$
$$\Gamma_{n,n-j,-,+} = \frac{\mathrm{i}^j(h_n - h'_{n'})}{2m} \sin\left(\tfrac{1}{2}\alpha_n\right) \sin\left(\tfrac{1}{2}\alpha'_{n'}\right) \; J^{l-1}_{-(j-1)}(1, \mathrm{i}, 0) \tag{A.21}$$

respectively.

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